STRUCTURE OF KÄHLER GROUPS, I : SECOND COHOMOLOGY

Alexander REZNIKOV

April 1, 1998

0. Introduction.

Fundamental groups of complex projective varieties are very difficult to understand. There is a tremendous gap between few computed examples and few general theorems. The latter all deal with either linear finite dimensional representations ([Sim]) or actions on trees ([Gr-Sch]); besides, one knows almost nothing.

This paper presents a new general theorem, partially settling a well-known conjecture of Carlson-Toledo (cf [Ko]).

Main Theorem. — *Let Γ be a fundamental group of a compact Kähler manifold. Assume Γ is not Kazhdan. Then $H^2(\Gamma, \mathbb{R}) \neq 0$. Morover, let Δ be a finitely presented group which is not Kazhdanand let $\Gamma \to \Delta$ be a central extension. If Γ is a fundamental group of a compact Kähler manifold, then the natural map in the second real cohomology $H^2(\Delta, \mathbb{R}) \to H^2(\Gamma, \mathbb{R})$ is nonzero.*

Corollary. — *Suppose Δ is not Kazhdan .Let $\widetilde{\Delta}$ be the universal central extension $0 \to H_2(\Delta, \mathbb{Z}) \to \widetilde{\Delta} \to \Delta \to 1$. Then Γ is not a fundamental group of a compact Kähler manifold.*

Examples.

1. Any amenable group is not Kazhdan, so their universal central extensions are not Kähler.

2. Any lattice in $SU(n, 1)$ is not Kazhdan. However, in this case one can do better, see remark in section 8 below. It follows from the Corollary to the Main Theorem that a universal central extension of such a lattice is not a Kähler group. In contrast with this, any central extension with finite cyclic group as a center,

Mots-clés : Kähler groups, property T.

whose extension class is a reduction *mod n* of the Kähler class *is* a Kähler group, as shown by Deligne, Kòllar and Catanese [Ko].

THEOREM 0.1. — *Let Γ be a fundamental group of a complex projective variety. Suppose Γ has a Zariski dense rigid representation in $SO(2,n)$, n odd. Then*

(i) $H^2(\Gamma,\mathbb{R}) \neq 0$.

(ii) Moreover, $H_b^2(\Gamma,\mathbb{R}) \neq 0$ *and the canonical map* $H_b^2 \to H^2$ *is not zero.*

COROLLARY 0.1. — *Let Δ be a lattice in $SO(2,n)$, n odd, uniform or not. Let $\widetilde{\Delta}$ be a universal central extension of Δ. Then Δ is not a fundamental group of a complex projective variety.*

THEOREM 0.2. — *Let Γ be a fundamental group of a complex projective variety. Suppose Γ has a Zariski dense rigid representation in $Sp(4)$. Then*

(i) $H^2(\Gamma,\mathbb{R}) \neq 0$.

(ii) Moreover, $H_b^2(\Gamma,\mathbb{R}) \neq 0$ *and the canonical map* $H_b^2 \to H^2$ *is not zero.*

COROLLARY 0.2. — *Let Δ be a lattice in $Sp(4)$. Then $\widetilde{\Delta}$ is not a fundamental group of a complex projective variety.*

COROLLARY 0.3. — *Lattices in* $\mathrm{Spin}(2,n)$, *n odd, and* $\widetilde{Sp(4)}$ *are not fundamental groups of a complex projective variety.*

The next result shows that three-manifold groups wich are rich in the sence of [Re2] are not Kähler .

THEOREM 10.3. — *(Rich Three-manifold groups are not Kähler). Let M^3 be an irreducible atoroidal three-manofold. Suppose there exists a Zariski dense homomorphism $\rho : \pi_1(M) \to SL_2(\mathbb{C})$. Then $\Gamma = \pi_1(M)$ is not Kähler.*

1. A geometric picture for rigid representations.

Let Y be a compact Kähler manifold. All rigid irreducible representations $\rho : \pi_1(Y) \to SL(N,\mathbb{C})$ are conjugate to representations landing in a $SU(m,n) \subset SL(N,\mathbb{C})$ with $m+n=N$ ([Sim 1]) and have a structure of complex variation of Hodge structure ([Sim 1]). Moreover, we can always arrange that this conjugate representation is defined over $\overline{\mathbb{Q}}$ (see e.g. [Re 1]). Relabeling , we assume that ρ itself is defined over $\overline{\mathbb{Q}}$. We assume that moreover, ρ is defined over $\mathcal{O}(\overline{\mathbb{Q}})$; by a conjecture of Carlos Simpson ([Sim 1]) this is always the case. Let $\{\rho_i\}$ be all the

Galois twists of ρ, then ρ_i are rigid therefore land in $SU(m_i, n_i)$. The image of $\pi_1(Y)$ in $\prod_i SU(m_i, n_i)$ is discrete; call it Γ.

Coming back to ρ, consider a corresponding θ-bundle E ([Sim 1]). It has the following structure: $E = \bigoplus_{p+q=k} L^{p,q}$ and θ maps $L^{p,q}$ to $L^{p-1,q+1} \otimes \Omega^1$. Any θ-invariant subbundle of E has negative degree; in particular, the degree of $L^{k,0}$ is positive. A harmonic metric K in E is the unique metric satisfying the equation ([Hit]) $F_K = [\theta, \theta^*]$. The hermitian connection ∇_K leaves all $E^{p,q}$ invariant. The connection $\nabla_K + \theta + \theta^*$ is flat with monodromy ρ. Let V be the corresponding flat holomorphic bundle. In V, we have a flag of holomorphic subbundles $F^p = V^{k,0} \oplus \cdots \oplus V^{p,k-p}$, where $V^{p,q}$ are $E^{p,q}$ thought as C^∞-subbundles of V with its new holomorphic structure. We have therefore a ρ-equivariant map $\widetilde{Y} \xrightarrow{s} D$, where D is a corresponding Griffiths domain ([G], ch. I–II). Changing the sign of K alternatively on $V^{p,q}$ we obtain a flat pseudo-hermitian metric in V.

So if $m = \bigoplus_{p \text{ even}} \dim V^{p,q}$, $n = \bigoplus_{p \text{ odd}} \dim V^{p,q}$, then ρ lands in $SU(m, n)$. The Griffiths domain D carries a horizontal distribution defined by the condition that the derivative of F_p lies in F_{p+1}. The developing map s is horizontal. Differentiating this condition we obtain a second order equation ([Sim 1]) $[\theta, \theta] = 0$, in other words for $Z, W \in T_xY$, $\theta(Z)$ and $\theta(W)$ commute.

Since the image of $\pi_1(Y)$ in $\prod_i SU(m_i, n_i)$ is discrete, we obtain

Proposition 1.1 (Geometric picture for rigid representations). — *Let $\rho : \pi_1(Y) \to SL(N, \mathbb{C})$ be a rigid irreducible representation, defined over $\mathcal{O}(\overline{\mathbb{Q}})$. Then there exist Griffiths domains $D_i = SU(m_i, n_i)/K_i$, a discrete group Γ in $\prod SU(m_i, n_i)$ and a horizontal holomorphic map*

$$S : Y \longrightarrow \prod D_i/\Gamma$$

which induces ρ and all its Galois twists.

Remark. — Though a Griffiths domain D is topologically a fibration over a hermitian symmetric space with fiber a flag variety, generally it does not have a $SU(m, n)$-invariant Kähler metric. So the complex manifold $\prod D_i/\Gamma$ is not Kähler.

Remark. — This proposition tells us that one cannot expect too many compact Kähler manifolds to have a nontrivial linear representation of their fundamental group, of finite dimension.

Lemma 1.2 (Superrigidity).

(1) Let $X = \Gamma \setminus SU(m,n)/S(U(m) \times U(n)$ be a compact hermitian locally symmetric space of Siegel type I. Let Y be compact Kähler and let $f : Y \to X$ be continuous. If $f_* : \pi_1(Y) \to SU(n,m)$ is rigid and Zariski dense, e.g. $f_* : \pi_1(Y) \to \Gamma$ an isomorphism, then either f is homotopic to a holomorphic map, or there exists a compact complex analytic space Y', $\dim Y' < \dim X$, and a holomorphic map $\varphi : Y \to Y'$ such that f is homotopic to a composition $Y \xrightarrow{\varphi} Y' \xrightarrow{f_1} X$.

(2) Let $X = \Gamma \setminus SO(2,n)/S(O(2) \times O(n))$ be of Siegel type IV. Let Y be compact Kähler and let $f : Y \to X$ be continuous. If $f_* : \pi_1(Y) \to SO(2,n)$ is rigid and Zariski dense then either f is homotopic to a holomorphic map f_0, or n is even and there exists a compact complex analytic space Y', $\dim Y' < \dim X$, and a holomorphic map $\varphi : Y \to Y'$ such that f is homotopic to a composition $Y \xrightarrow{\varphi} Y' \xrightarrow{f_1} X$.

(3) Let $X = \Gamma \setminus Sp(2n)/U(n)$ be of Siegel type III. Let Y be compact Kähler and let $f : Y \to X$ be continuous. If $f_* : \pi_1(Y) \to Sp(2n)$ is rigid and Zariski dense, then either f is homotopic to a holomorphic map f_0, or there exists a compact complex analytic space Y', $\dim Y' < \dim Y$, and a holomorphic map $\varphi : Y \to Y'$, such that f is homotopic to a composition $Y \xrightarrow{\varphi} Y' \xrightarrow{f_1} X$.

(4) Let $X = \Gamma \setminus Sp(4)/U(2)$ be a Shimura threefold. Let Y be compact Kähler and let $f : Y \to X$ be continuous. Then either f is homotopic to a holomorphic map f_0, or there exists a (singular) proper curve S, and a holomorphic map $\varphi : Y \to S$, such that f is homotopic to a composition $Y \xrightarrow{\varphi} S \longrightarrow X$.

Remarks.

1. I leave the case of Siegel type II to the reader (the proof is similar).

2. If f_* is *not* rigid, one has strong consequences for $\pi_1(Y)$, see 9.1.

3. The lemma should be viewed as a final (twistorial) version of the superrigidity theorem ([Si]).

2. Proof of the Superrigidity Lemma (1)**.**

(1) Since we are given a continuous map $Y \xrightarrow{f} X = \Gamma \setminus SU(m,n)/S(U(m) \times U(n))$ the map S of Proposition 1.1 is simply a holomorphic map $Y \to D/\Gamma$, where D is a Griffiths domain corresponding to the complex variation of Hodge structure, defined by $\rho = f_* : \pi_1(Y) \to SU(m,n)$. Suppose the Higgs bundle looks like $\oplus E^{p,q}$ where $E^{p,q}$ have dimensions $m_1, n_1, m_2, n_2, \ldots, m_s, k_s$ where k_s is possibly missing. Then $\sum m_i = m$, $\sum n_i = n$. Now, the dimension of the horizontal distribution is

$$m_1 \cdot n_1 + n_1 \cdot m_2 + m_2 \cdot n_2 + \cdots + m_s \cdot k_s \ .$$

We notice that this number is strictly less than $m \cdot n = \dim X$ except for the cases:

I) $s = 1$, *i.e.* $E = E^{1,0} \oplus E^{0,1}$

II) $s = 2$, $k_2 = 0$, *i.e.* $E = E^{2,0} \oplus E^{1,1} \oplus E^{0,2}$.

In the first case, D is the symmetric space, and $D/\Gamma = X$ so we arrive to a holomorphic map to. $Y \to X$. In the second case the second order equation reads $\theta_1(Z)\theta_2(W) - \theta_1(W)\theta_2(Z) = 0$ where $\theta_1 : TY \otimes E^{0,2} \to E^{1,1}$ and $\theta_2 : TY \otimes E^{1,1} \to E^{2,0}$ are the components of the (horizontal) derivative DS. So the image of DS is strictly less than $\mathrm{Hom}(E^{0,2}, E^{1,1}) \oplus \mathrm{Hom}(E^{1,1}, E^{1,0}) = m \cdot n = \dim X$. In other words, $\dim Y' < \dim X$ where $Y' = S(Y)$.

3. Variation of Hodge structure, corresponding to rigid representations to $SO(2, n)$.

Let $\rho : \pi_1(Y) \to SO(2, n)$ be a Zariski dense rigid representation. Complexifying, we obtain a variation of Hodge structure $E = \oplus E^{p,q}$. Since ρ is defined over reals, we deal with real variation of Hodge structure ([Sim 1]) that is to say, $E^{p,q} = \overline{E^{p,q}}$ with respect to a flat complex conjugation. For $n \geq 3$, this leaves exactly two possibilities:

I) $E = E^{2,0} \oplus E^{1,1} \oplus E^{0,2}$, $\dim E^{1,1} = n$, $\dim E^{2,0} = \dim E^{0,2} = 1$.

II) n is even, $E = E^{2,0} \oplus E^{1,1} \oplus E^{0,2}$, $\dim E^{1,1} = 2$, $\dim E^{2,0} = \dim E^{0,2} = n/2$.

In case I) the Griffits domain is the symmetric spaces $SO(2, n)/S(O(2) \times O(n))$ so the harmonic metric viewed as a harmonic section of the flat bundle with fiber a symmetric space, is holomorphic. In the second case the second order equation implies that the rank of the derivative DS of the ρ-equivariant holomorphic map $\widetilde{Y} \to D$ is strictly less than n.

Proof of the Superrigidity Lemma (2).

This follows immediately from the previous discussion in the same manner as in (1)

4. Variations of Hodge structure, corresponding to rigid representation to $Sp(4)$.

Let $\rho : \pi_1(Y) \to Sp(2n)$ be a Zariski dense rigid representation. Complexifying, we obtain a representation $\pi : \pi_1(Y) \to SU(n, n)$ and a real variation of Hodge structure $E = \bigoplus_{p+q=k} E^{p,q}$, $E^{p,q} = \overline{E}^{q,p}$ and k odd. For $n = 2$ this leaves two possibilities:

(I) $E = E^{1,0} \oplus E^{0,1}$, and both $E^{1,0}$ and $E^{0,1}$, or rather $V^{1,0}$ and $V^{0,1}$ viewed as C^∞-subbundles of the flat bundle V, are lagrangian wi th respect to the flat complex symplectic structure. This means first, that the Griffiths domain D is the symmetric space $SU(2,2)/S(U(2) \times U(2)$, second, that the image of the equivariant horizontal holomorphic map $S : \widetilde{Y} \to D$ lies in the copy of the Siegel upper half-space $Sp(4)/U(2)$ under the Satake embedding ([Sa]). In other words, the unique ρ-equivariant harmonic map $\widetilde{Y} \to Sp(4)/U(2)$ is holomorphic.

II) n is even, $E = E^{2,0} \oplus E^{1,1} \oplus E^{0,2}$, $\dim E^{1,1} = 2$, $\dim E^{2,0} = \dim E^{0,2} = n/2$.

In case I) the Griffits domain is the symmetric spaces $SO(2,n)/S(O(2) \times O(n))$ so the harmonic metric viewed as a harmonic section of the flat bundle with fiber a symmetric space, is holomorphic. In the second case the second order equation implies that the rank of the derivative DS of the ρ-equivariant holomorphic map $\widetilde{Y} \to D$ is strictly less than n.

Proof of the Superrigidity Lemma (2)**.**

This follows immediately from the previous discussion in the same manner as in (1)

4. Variations of Hodge structure, corresponding to rigid representation to $Sp(4)$.

Let $\rho : \pi_1(Y) \to Sp(2n)$ be a Zariski dense rigid representation. Complexifying, we obtain a representation $\pi : \pi_1(Y) \to SU(n,n)$ and a real variation of Hodge structure $E = \bigoplus_{p+q=k} E^{p,q}$, $E^{p,q} = \overline{E}^{q,p}$ and k odd. For $n = 2$ this leaves two possibilities:

(I) $E = E^{1,0} \oplus E^{0,1}$, and both $E^{1,0}$ and $E^{0,1}$, or rather $V^{1,0}$ and $V^{0,1}$ viewed as C^∞-subbundles of the flat bundle V, are lagrangian with respect to the flat complex symplectic structure. This means first, that the Griffiths domain D is the symmetric space $SU(2,2)/S(U(2) \times U(2)$, second, that the image of the equivariant horizontal holomorphic map $S : \widetilde{Y} \to D$ lies in the copy of the Siegel upper half-space $Sp(4)/U(2)$ under the Satake embedding ([Sa]). In other words, the unique ρ-equivariant harmonic map $\widetilde{Y} \to Sp(4)/U(2)$ is holomorphic.

(II) $E = E^{3,0} \oplus E^{2,1} \oplus E^{1,2} \oplus E^{0,3}$ and $\dim E^{p,q} = 1$. The second order equation for θ implies immediately that D_s has rank at most one everywhere on Y.

Proof of the Superrigidity Lemma (4)**.**

Follows from the discussion above.

6. Variations of Hodge structure, corresponding to rigid representation to $Sp(2n)$, and proof of the Superrigidity Lemma (3).

In general, the Higgs bundle is $E = \bigoplus\limits_{p+q=2s+1} E^{p,q}$, $E^{p,q} = \overline{E^{p,q}}$. The dimension of the horizontal distribution is

$$d = \sum_{p<s} \dim E^{p,q} \cdot \dim E^{p+1,q-1} = \frac{\dim E^{s,s+1} \cdot (\dim E^{s,s+1} + 1)}{2} \ ,$$

since $\theta : E^{s,s+1} \to E^{s+1,s}$ viewed as bilinear form, should be symmetric. Moreover, $\sum\limits_{p\leq s} \dim E^{p,q} = n$. An elementary exercise shows that if $s > 1$, $d < \frac{n(n+1)}{2}$. If $s = 1$, we get a holomorphic map to the Siegel upper half-plane.

7. Regulators, I: proof of the Main Theorem.

The reader is supposed to be familiar with the geometric theory of regulators ([Re 1], [Co]).

Let $\mathbb{H}$ be a complex Hilbert space. The constant Kähler form (dX, dX) is invariant under the affine isometry group $\mathrm{Iso}(\mathbb{H})$, and $\mathbb{H}$ is contractible, therefore there is a regulator class in $H^2(\mathrm{Iso}^{\delta}(\mathbb{H}), \mathbb{R})$. In fact, there is a class ℓ in $H^1(\mathrm{Iso}^{\delta}(\mathbb{H}), \mathbb{H})$ defined by a cochain $(x \mapsto Ux + b) \longmapsto b$. The regulator class is simply (ℓ, ℓ).

If $\pi_1(Y)$ does not have property T, then there exist a representation $\rho : \pi_1(Y) \to \mathrm{Iso}(\mathbb{H})$ and a holomorphic nonconstant section S of the associated flat holomorphic affine bundle with fiber $\mathbb{H}$ ([Ko-Sch]). It follows that the pull-back $\rho^*((\ell, \ell))$ of the regulator class to $H^2(\pi_1(Y), \mathbb{R})$ restricts to a cohomology class in $H^2(Y, \mathbb{R})$, given by a non-zero semi-positive $(1,1)$ form. Multiplying by the ω^{n-1}, where ω is a Kähler form, and $n = \dim Y$, and integrating over Y we get a positive number, therefore this cohomology class is non-zero. Therefore $H^2(\pi_1(Y), \mathbb{R}) \neq 0$.

Now if Δ does not have property T, and if $\pi_1(Y) \to \Delta$ is a central extension, then the construction of [Ko-Sch] gives us an isometric uniform action on a real Hilbert space $\pi_1(Y) \to \mathrm{Iso}(\mathbb{H})$, which factors through Δ, an d a harmonic section of an associated flat bundle. Since the action is uniform, the corresponding linear representation $\rho : \Delta \to U(\mathbb{H})$ does not have fixed vectors. It follows from the Lyndon-Serre-Hochschild spectral sequence that the map $H^1(\Delta, \mathbb{H}) \to H^1(\pi_1(Y), \mathbb{H})$ is an isomorphism. By [Ko-Sch], thee exists an isometric action of $\pi_1(Y)$ on the complexified Hilbert space, extending the previous one, such that the corresponding

flat bundle has a holomorphic section. This act ion nesessarily factors through Δ. Arguing as above, we deduce the theorem.

Remark. — Historically, the first break through in this direction has been made in [JR], under assumption of having a nontrivial variation of a finite-dimensional unitary representation. Compare Proposition 9.1 below.

Proof of the Corollary 0.2. — The Lyndon-Serre-Hochschild spectral sequence implies that the map $H^2(\Delta,\mathbb{R}) \to H^2(\widetilde{\Delta},\mathbb{R})$ is zero. So $\widetilde{\Delta}$ is not a Kähler group.

Remark. — Suppose $\pi_1(Y)$ does not have property T. Suppose moreover that that $\pi_1(Y)$ has a permutation representation in $\ell^2(B)$, where B is a countable set, and $H^1(\pi_1(Y),\ell^2(B)) \neq 0$. Then we actually proved that $H^2(\pi_1(Y),\ell^1(B)) \neq 0$. That is because the scalar product $\ell^2(B)\times\ell^2(B) \to \mathbb{C}$ factors through $\ell^1(B)$. Moreover, the canonical map $H^2(\pi_1(Y),\ell^1(B)) \to H^2(\pi_1(Y),\mathbb{C})$ is nonzero.

8. Regulators, II: proof of Theorems 0.1, 0.2.

Let G be an isometry group of a classical symmetric bounded domain D. With the exception of $SO(2,2)$, $H^1(G,\mathbb{Z}) = \mathbb{Z}$. This defines a central extension $1 \to \mathbb{Z} \to \widetilde{G} \to G \to 1$ and an extension class $e \in H^2(G^\delta,\mathbb{Z})$. On the other hand, the Bergman metric on D is G-invariant, so it defines a regulator class $r \in H^2_{\mathrm{cont}}(G,\mathbb{R})$. It is proved in [Re 2], [Re 3] that, first, these classes coincide up to a factor, and second, lie in the image of the bounded cohomology: $H^2_b(G^\delta,\mathbb{R}) \to H^2(G^\delta,\mathbb{R})$.

If Y is a compact Kähler manifold, $\rho : \pi_1(Y) \to G$ a representation , s a holomorphic nonconstant section of the associated flat D-bundle, then one sees immediately that $(\rho^*(r),\omega^{n-1}) > 0$, so $\rho^*(r)$, $\rho^*(e) \neq 0$. Theorem 0.1 follows now from the analysis of VHS given in sections 3, 4. To prove Theorem 0.2 notice that the case when Y fibers over a curve is obvious, otherwise Y admits a holomorphic map to a quotient of the Siegel half-plane and the proof proceeds as before.

Remark. — By [CT 1], the result of Theorem 0.1 is true for lattices in $SU(n,1)$.

9. Nonrigid representations.

PROPOSITION 9.1. — *Let Y be compact Kähler and let $\rho : \pi_1(Y) \to SL(n,\mathbb{C})$ be a nonrigid irreducible representation. Then $H^2(\pi_1(Y),\mathbb{R}) \neq 0$.*

Proof. — Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and let $\bar{\rho}$ be the adjoint representation. We know that $H^1(\pi_1(Y), \mathfrak{g}) \neq 0$. Therefore $H^1(Y, \underline{\mathfrak{g}}) \neq 0$ where $\underline{\mathfrak{g}}$ is the local system. By the Simpson's hard Lefschetz ([Sim 1]), the multiplication by ω^{n-1} gives an isomorphism $H^1(Y, \underline{\mathfrak{g}}) \to H^{2n-1}(Y, \underline{\mathfrak{g}})$, where ω is the polarization class and $n = \dim_{\mathbb{C}} Y$. The Poincaré duality implies that the Goldman's pairing $H^1(Y, \mathfrak{g}) \times H^1(Y, \mathfrak{g}) \to \mathbb{C}$ is nondegenerate. Let z be homology class in $H_2(Y)$, dual to ω^{n-1}, and $\bar{z}$ its image in $H_2(\pi_1(Y))$. It follows that the pairing $H^1(\pi_1(Y), \mathfrak{g}) \times H^1(\pi_1(Y), \mathfrak{g}) \to \mathbb{C}$ defined by $f, g \mapsto [(f, g), \bar{z}]$ is nondegenerate. Here $(f, g) \in H^2(\pi_1, \mathbb{C})$ is the pairing defined by the Cartan-Killing form. In particular, $\bar{z} \neq 0$.

Corollary 10.1. — *Let Y a compact Kähler manifold. If $\pi_1(Y)$ has a Zariski dense representation in either $Sp(4)$ or $SO(2, n)$, n odd, then $H^2(\pi_1(Y), \mathbb{R}) \neq 0$.*

Proof. — For rigid representations, this is proved in Theorems 0.1, 0.2. For nonrigid representations, this follows from Proposition 9.1.

Corollary 9.2. — *Let Γ be any overgroup of a Zariski dense countable subgroup of $Sp(4)$ or $SO(2, n)$, n odd. Suppose $b_1(\Gamma) = 0$. Then the universal central extensions $\widetilde{\Gamma}$ is not Kähler.*

10. Three-manifolds groups are not Kähler.

In this section, based on the previous development, we will present a strong evidence in favour of the following conjecture, which we formulated in 1993 (Domingo Toledo informs us that a similar conjecture had been discussed by Goldman and Donaldson in 1989):

Conjecture 10.1. — *Let M^3 be irreducible closed 3-manifold with $\Gamma = \pi_1(M)$ infinite. Then Γ is not Kähler.*

Proposition 10.2 (Seifert fibration case). — *A cocompact lattice in $\widetilde{SL(2, \mathbb{R})}$ is not Kähler.*

Proof. — Passing to a subgroup of finite index, we can assume that Γ is a central extension of a surface group:

$$1 \longrightarrow \mathbb{Z} \longrightarrow \Gamma \longrightarrow \pi_1(S) \longrightarrow 1$$

with a nontrivial extension class. In particular, $H^1(\Gamma, \mathbb{Q}) \simeq H^1(\pi_1(S), \mathbb{Q}))$, so the multiplication in $H^1(\Gamma, \mathbb{Q})$ is zero, which is impossible if Γ is Kähler.

Recall that "most" of closed three-manifolds admit a Zariski dense homomorphism $\pi_1(M) \xrightarrow{\rho} SL_2(\mathbb{C})$ ([CGLS], [Re 1]).

THEOREM 10.3. — *Let M^3 be atoroidal. Suppose there exists a Zariski dense homomorphism $\rho : \pi_1(M) \to SL_2(\mathbb{C})$. Then $\Gamma = \pi_1(M)$ is not Kähler.*

Proof. — By a theorem of [Zi] $\pi_1(M)$ does not have property T. By the Main Theorem, $H^2(\Gamma, \mathbb{R}) \neq 0$, hence by [Th], M is hyperbolic, which is impossible by [CT 1].

Alternatively, ρ is not rigid by [Sim 1], so $H^2(\Gamma, \mathbb{R}) \neq 0$ by Proposition 9.1, and then one procedes as before.

Remark. — In view of [CGLS], [Re 1], we obtain a huge number of groups which are not Kähler.

11. Central extensions of lattices in $PSU(2,1)$.

We saw a general result, that, if $\Gamma \subset SU(n,1)$ a cocompact lattice and $[\omega] \in H^2(\Gamma, \mathbb{Z})$ is given by any ample line bundle, then a central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\Gamma} \longrightarrow \Gamma \longrightarrow 1$$

with the extension class $[\omega]$ is not Kähler. For $n = 2$ one can also prove:

THEOREM 11.1. — *Let $\omega \in H^2(B^2/\Gamma, \mathbb{Z}) \cap (H^{2,0} \oplus H^{0,2})$, $\omega \neq 0$. Then an extension*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\Gamma}_\omega \longrightarrow \Gamma \longrightarrow 1$$

with the extension class ω is not Kähler.

Remark. — $H^{2,0}$ becomes big on étale finite coverings of B^2/Γ by Riemann-Roch.

Proof. — Suppose $\widetilde{\Gamma}_\omega = \pi_1(Y)$. The representation $\pi_1(Y) \to \Gamma \to SU(2,1)$ is rigid by the Lyndon-Serre-Hochschild spectral sequence. It follows that there exists a dominating holomorphic map $Y \to B^2/\Gamma$. But then the pullback map on $H^{2,0}$ is injective, a contradiction.

12. Smooth hypersurfaces in ball quotients which are not $K(\pi,1)$.

We saw that under various algebraic assumptions on $\Gamma = \pi_1(Y)$, there is a class in $H^2(Y,\mathbb{R})$ which vanishes on the Hurewitz image $\pi_2(Y) \to H_2(Y,\mathbb{Z})$, therefore defining a nontrivial element of $H^2(\Gamma,\mathbb{R})$. On the contrary, we will show now that there are hypersurfaces Y in ball quotients B^n/Γ, $n \geq 3$ with a surjective map $\pi_1(Y) \to \Gamma$ such that $\pi_i(Y) \neq 0$ for some i. The proof is very indirect and we don't know the exact value of i. The varieties Y were in fact introduced in [To] where it is proved that $\pi_1(Y)$ is not residually finite. We will show that $cd(\pi_1(Y)) \geq 2n-1$, therefore Y is not $K(\pi,1)$.

Let X^n be an arithmetic ball quotient and let $X_0 \subset X$ be a totally geodesic smooth hypersurface. Let $D = X - X_0$, then D is covered topologically by $\mathbb{C}^n$ minus a countable union of hyperplanes, so D is $K(\pi,1)$. Let S be a boundary of a regular heighbourhood of X_0, so S is a circle bundle over X_0, in particular S is $K(\pi,1)$ and $\pi_1(S)$ is a central extension $0 \to \mathbb{Z} \to \pi_1(S) \to \pi_1(X_0) \to 1$ with a nontrivial extension class (this is because the normal bundle to X_0 is negative). Let V be a finite dimensional module over $\pi_1(X)$ with an invariant nondegenerate form $V \to V'$. We have an exact sequence

$$H^{2n-1}(\pi_1(X),V) \longrightarrow H^{2n-1}(\pi_1(X_0),V) \oplus H^{2n-1}(\pi_1(D),V) \longrightarrow H^{2n-1}(\pi_1(S),V) \longrightarrow H^{2n}(\pi_1(X),V) \longrightarrow \cdots$$

Now, we make a first assumption:

1) $H^0(\pi_1(X),V) = 0$.

It follows that $H_0(\pi_1(X),V) = 0$, so $H^{2n}(\pi_1(X),V) = 0$; we make a second assumption :

2) $H^1(\pi_1(X),V) = 0$.

It follows that $H^{2n-1}(\pi_1(X),V) = 0$. So we have (remember that X_0 has dimension $n-1$)

$$H^{2n-1}(\pi_1(D),V) \simeq H^{2n-1}(\pi_1(S),V) \ .$$

Now, in the E^2 of the Lyndon-Serre-Hochschild spectral sequence for $H^*(\pi_1(S),V)$ the term $H^{2n-2}(\pi_1(X_0),H^1(\mathbb{Z},V))$ is not hit by any differential. Since $\mathbb{Z}$ acts trivially, this is just $H^{2n-2}(\pi_1(X_0),V) \simeq H_0(\pi_1(X_0),V)$. We now make a third assumption:

3) $H^0(\pi_1(X_0),V) \neq 0$.

Then we will have $H^{2n-1}(\pi_1(D),V) \neq 0$. Let Y be a generic hyperplane section of X/X_0, constructed in [To], then [GM], $\pi_1(Y) = \pi_1(D)$ and we are done.

Now, we take for V the adjoint module. The assumption 2) follows from Weil's rigidity. The assumption 3) is satisfied for standard examples of X_0 ([To]).

Remark. — The construction of [To] is given for lattices in $SO(2,n)$, but it applies verbatim here.

References

[Co] K. Corlette. — *Rigid representations of Kählerian fundamental groups*, J.D.G. **33** (1991), 239–252.

[CT 1] J. Carlson, D. Toledo. — *Harmonic maps of Kähler manifolds to locally symmetric spaces*, Publ. IHES **69** (1989), 173–201.

[CGLS] M. Culler C. Gordon, J. Luecke, P. Shalen. — *Dehn surgery on knots*, Annals of Math. **125** (1987), 237–308.

[G] Ph. Griffiths, ed.. — *Topics in transcendental algebraic geometry*, Princeton UP, 1984.

[Gr-Sch] M. Gromov, R. Schoen. — *Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one*, Publ. Math. Inst. Hautes Étud. Sci. **76** (1992), 165–246.

[GS] Ph. Griffiths, W. Schmidt. — *Locally homogeneous complex manifolds*, Acta Math. **123** (1969), 253–302.

[H] P. de la Harpe, A.Valette. — *La propriété (T) de kazhdan pour les groupes localement compactes*, Astérisque, 1989.

[Hit] N. Hitchin. — *The self-duality equation on a Riemann surfaces*, Proc. London Math. Soc. **55 (3)** (1987), 59–126.

[JR] F.E.A.Johnson, E.G.Rees. — *On the fundamental group of a complex algebraic manifold*, Bull. Lond. Math. Soc. **19** (1987), 463–466.

[Ko] J. Kòllar. — *Shafarevich maps and automorphic forms*, Princeton UP, 1995.

[Ko-Sch] N. Korevaar, R. Schoen. — *Global existence theorems for harmonic maps to non locally compact spaces*, Commun. Anal. Geom. **5 (2)** (1997), 333–387.

[Re 1] A. Reznikov. — *Rationality of secondary classes*, J. Differential Geom. **43** (1996), 674–692.

[Re 2] A. Reznikov. — *Three-manifolds class field theory*, Selecta Math. **3** (1997), 361–399.

[Sa] I. Satake. — *Algebraic structures of symmetric domains*, Princeton UP, 1980.

[Sim 1] C. Simpson. — *Higgs bundles and local systems*, Publ. IHES **75** (1992), 5–95.

[Sim 2] C. Simpson. — *Moduli of representations of the fundamental group of smooth projective variety*, Publ. IHES **79-80** (1994), 47–129, 5–79.

[Si] Y.-T. Siu. — *The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds*, Ann. Math. **112** (1980), 73–111.

[Th] W. Thurston. — *The geometry and topology of three manifolds*, Lecture notes, Princeton University.

[To] D. Toledo. — *Projective varieties with non-residually finite fundamental group*, Publ. IHES **77** (1993), 103–119.

[Zi] R. Zimmer. — *Kazhdan groups acting on compact manifolds*, Invent. Math. **75** (1984), 425–436.

–◊–

Department of Mathematical Sciences
University of Durham
South Road
Durham DH1 3LE(UK)

Current address :
Université de Grenoble I

Institut Fourier
UMR 5582
UFR de Mathématiques
B.P. 74
38402 St MARTIN D'HÈRES Cedex (France)

Alexander.Reznikov@durham.ac.uk